Task Set Synthesis with Cost Minimization for Sporadic Real-Time Tasks

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Outline

Introduction

Task Set Synthesis Problem

Proposed Combinatorial Algorithms

Conclusion
Decomposition of the Analysis

- Typical analysis and optimizations in real-time systems are decomposed into two phases
  - Phase 1: Worst-case execution time (WCET) of a stand-alone program
    - using WCET analyzers such as aiT or Chronous.
  - Phase 2: Worst-case response time of a periodic/sporadic task by considering the competition with the other tasks
    - analyzing the worst-case interference from the other tasks
    - many techniques such as utilization-based tests, response time analysis, busy-interval techniques, real-time calculus, max-plus algebra, etc.
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Sporadic Task Models

Sporadic Task $\tau_i$:

- $T_i$ is the minimal time between any two consecutive job releases
- A relative deadline $D_i$ for each job from task $\tau_i$
- $(C_i, T_i, D_i)$ is the specification of sporadic task $\tau_i$, where $C_i$ is the worst-case execution time.

- implicit deadline: $D_i = T_i$, for every task $\tau_i$,
- constrained deadline: $D_i \leq T_i$, for every task $\tau_i$
- arbitrary deadline: otherwise
Cost-Dependent WCET

- Deriving WCET is not a simple problem
- By spending more cost, the WCET may be reduced
  - Using more SRAM in the system or larger cache size
  - Using code redundancy or execution reordering to improve the reliability
- By reducing the quality of computation, the WCET may be reduced
  - Imprecise computation
  - Multiple versions of execution plans with different qualities

QRAM Model (Rajkumar et al. RTSS’97)

QRAM model: maximizing the system quality by choosing proper versions to meet the timing constraints of real-time tasks.
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Minimum Cost Synthesis Problem

Input:
- A sporadic real-time task set \( \mathcal{T} \)
- Each task \( \tau_i \in \mathcal{T} \) has
  - \( T_i \): minimum inter-arrival time
  - \( D_i \): relative deadline
- \( \tau_i \) has \( w_i \geq 1 \) different versions with different costs
  - \( \theta_i(k) \) is the cost for the \( k \)-th version of task \( \tau_i \)
  - \( C_i^{\theta_i(k)} \) is the corresponding WCET
  - \( U_i^{\theta_i(k)} = \frac{C_i^{\theta_i(k)}}{T_i} \) as the utilization
- Without loss of generality, \( \theta_i(1) < \theta_i(2) < \cdots < \theta_i(w_i) \)

Output:
Select one version \( m_i \) for task \( \tau_i \) such that \( \mathcal{T} \) be feasibly scheduled and the system cost \( \sum_{\tau_i \in \mathcal{T}} \theta_i(m_i) \) is minimized.
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Schedulability of EDF

- Implicit deadlines: EDF is feasible if and only if the total utilization $U = \sum_{\tau_i \in T} \frac{C_i}{T_i}$ is at most 100%.
- Constrained/arbitrary deadlines: demand bound testing is required.

Baruah et al. [RTSS 1990]: A task set $T$ can be feasibly scheduled (under EDF) on one processor if and only if

$$\forall t \geq 0, \sum_{\tau_i \in T} dbf(\tau_i, t) = \sum_{i=1}^{n} \left[ \frac{t + T_i - D_i}{T_i} \right] C_i \leq t.$$
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Baruah et al. [RTSS 1990]: A task set $\mathcal{T}$ can be feasibly scheduled (under EDF) on one processor if and only if

$$\forall t \geq 0, \sum_{\tau_i \in \mathcal{T}} \text{dbf}(\tau_i, t) = \sum_{i=1}^{n} \left[ t + \frac{T_i - D_i}{T_i} \right] C_i \leq t.$$
Rate-Monotonic (RM) Scheduling

Priority Definition: A task with a smaller period has higher priority, in which ties are broken arbitrarily, i.e., $T_i \leq T_j$ if $i \leq j$.

- Least utilization upper bound for implicit deadlines:
  \[ U = \sum_{\tau_i \in \mathcal{T}} \frac{C_i}{T_i} \leq n(2^{\frac{1}{n}} - 1) \text{ for } n \text{ tasks} \]

- If the following condition holds, the task set is schedulable under RM:
  \[ \forall \tau_i \in \mathcal{T} \exists t \text{ with } 0 < t \leq D_i \text{ and } W_i(t) \leq t, \]

where $W_i(t)$ of task $\tau_i$ is defined as follows:

\[ W_i(t) = C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j. \]
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Optimality of RM and EDF

- For uniprocessor scheduling, if there exists a feasible schedule, scheduling jobs by using EDF is also feasible.
  - EDF scheduling algorithm is optimal
- If a set of n implicit-deadline tasks, can be feasibly scheduled on a processor with a fixed-priority assignment, assigning tasks by using rate monotonic scheduling also leads to a feasible schedule.
  - RM scheduling algorithm is optimal for fixed-priority scheduling
  - Deadline Monotonic (DM) scheduling algorithm is optimal for fixed-priority scheduling
The issue for uniprocessor scheduling is on how to analyze the schedulability.

Verifying the schedulability is \( \mathcal{NP} \)-hard or \( \mathsf{coNP} \)-hard.

Approximations are possible, but what do we approximate when only binary decisions (Yes or No) have to be made?

- Answers like probabilistic guarantee are unlikely preferred, e.g., the task set is 99% schedulable.
- Resource augmentation by \( \rho \) requires a faster platform.
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Schedulability

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Resource Augmentation

For an algorithm A with a $\beta$ resource augmentation factor, it guarantees that

⇒

if the task set (system) is schedulable (feasible), Algorithm A will also returns a schedulable (feasible) answer by speeding up the system by a factor $\beta$, or

⇐

if Algorithm A does not return a schedulable (feasible) answer, the system is also unschedulable (infeasible) by slowing down by a factor $\beta$. 

Resource Augmentation

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Time Demand Function Revisit for RM

Let $w_i(t)$ of the task $\tau_i$ be defined as follows

$$w_i(t) = \left\lceil \frac{t}{T_i} \right\rceil C_i.$$  

### Graphical Representation

![Graph Illustrating $w_i(t)$](image)

$$W_i(t) = C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j$$

Schedulable if for each $\tau_i$ $\exists t$ with $W_i(t) \leq t$. 

11 06,12,2013, in RTSS’13, Vancouver, Canada, by Dr. Jian-Jia Chen
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The linear approximation makes the schedulability test easier:
- The test can be done in $O(n^2)$
- The resource augmentation factor is 2. [Albers and Slomka ECRTS’04]

\[ W_i(t) = C_i + \sum_{j=1}^{i-1} \left\lfloor \frac{t}{T_j} \right\rfloor C_j \]

Schedulable if for each $\tau_i \exists t$ with $W_i(t) \leq t$. 

\[ w_i(t) \leq w_i^*(t) \leq 2w_i(t) \]
Define demand bound function \( \text{dbf}(\tau_i, t) \) as

\[
\text{dbf}(\tau_i, t) = \max \left\{ 0, \left\lfloor \frac{t + T_i - D_i}{T_i} \right\rfloor \right\}
\]

\[
C_i = \max \left\{ 0, \left\lfloor \frac{t - D_i}{T_i} \right\rfloor + 1 \right\}
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Define demand bound function $\text{dbf}(\tau_i, t)$ as

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\text{dbf}(\tau_i, t) = \max \left\{ 0, \left\lfloor \frac{t + T_i - D_i}{T_i} \right\rfloor \right\} \quad C_i = \max \left\{ 0, \left\lfloor \frac{t - D_i}{T_i} \right\rfloor + 1 \right\}
$$

The diagram illustrates the relationship between $\text{dbf}(\tau_i, t)$ and $\text{dbf}^*(\tau_i, t)$, with $\text{dbf}(\tau_i, t) \leq \text{dbf}^*(\tau_i, t) \leq 2\text{dbf}(\tau_i, t)$.
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- The linear approximation makes the schedulability test easier
  - The test can be done in \( O(n^2) \)
  - The resource augmentation factor is 1.6322. [Chen and Chakraborty, RTSS’11]
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- Without loss of generality, $\theta_i(1) < \theta_i(2) < \cdots < \theta_i(w_i)$

Output:
Select one version $m_i$ for task $\tau_i$ such that $\mathcal{T}$ be feasibly scheduled and the system cost $\sum_{\tau_i \in \mathcal{T}} \theta_i(m_i)$ is minimized.
The utilization bound 100% is a necessary and sufficient test.

The problem is equivalent to the minimum multiple-choice knapsack problem.

Given a set of items, each with \( w_i \) versions and each version has a weight and a value, the objective is to choose one version in each item such that the total weight is no more than a given limit and the total value is as small as possible.

Many results are already known.

(α, β)-Approximation

- Suppose the optimal system cost is $B^*(I)$ for an input instance $I$.
- An algorithm has an $α$-approximation if it guarantees to have at most $α \cdot B^*(I)$ system cost for any input instance $I$.
- An $(α, β)$-approximation guarantees to have at most $α \cdot B^*(I)$ system cost by using $β$ speed augmentation factor.
  - With respect to speeding-up: the derived solution is a feasible solution by speeding up the platform to $β$, and has an $α$-approximation in the system cost with respect to the original instance.
  - With respect to slowing-down: the derived solution is $α$-approximation with respect to a problem instance by slowing down the platform to $\frac{1}{β}$.

An optimal algorithm for the minimum multi-choice knapsack problem:
- $(1, 1)$ for EDF with implicit deadlines
- $(1, \frac{1}{\ln 2})$ for RM with implicit deadlines
(\alpha, \beta)-\text{Approximation}

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- An \((\alpha, \beta)\)-approximation guarantees to have at most \( \alpha \cdot B^*(I) \) system cost by using \( \beta \) speed augmentation factor.
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An optimal algorithm for the minimum multi-choice knapsack problem:

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Theorem

Unless $P = NP$, there is no polynomial-time $(\alpha, 1)$-approximation algorithm for the minimum cost synthesis problem for any $\alpha \geq 1$ when considering EDF or FP scheduling.

Proof

It is based on an L-reduction from the uniprocessor schedulability problem for sporadic real-time tasks:

- Each task has two versions
- The one with cost equals to 1 has small execution time, and another one is with “very high” cost with 50% reduction of the execution time.
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Deadline Monotonic (DM) is optimal when $D_i \leq T_i$. 

- $C_1^1 = 1, C_1^2 = 0.5, C_1^3 = 0.25, T_1 = 2, D_1 = 2$.  
- $C_2^1 = 2, C_2^2 = 1.5, C_2^3 = 1, T_2 = 8, D_2 = 6$. 

Almost all the equations are different from the paper for simplicity.
DM Scheduling for Constrained Deadlines

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Deadline Monotonic (DM) is optimal when \( D_i \leq T_i \).

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\begin{align*}
C_1^1 &= 1, & C_1^2 &= 0.5, & C_1^3 &= 0.25, & T_1 &= 2, & D_1 &= 2. \\
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DM Scheduling for Constrained Deadlines

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Deadline Monotonic (DM) is optimal when $D_i \leq T_i$.

\[ w_1^3(6) = 1 \quad w_2^1(6) = 3.5 \quad w_2^2(6) = 2.625 \quad w_3^2(6) = 1.75 \]

- $C_1^1 = 1$, $C_1^2 = 0.5$, $C_1^3 = 0.25$, $T_1 = 2$, $D_1 = 2$.
- $C_2^1 = 2$, $C_2^2 = 1.5$, $C_2^3 = 1$, $T_2 = 8$, $D_2 = 6$. 
Schedulability for DM

Deadline Monotonic (DM) is optimal when \( D_i \leq T_i \).

For a given selection of versions \((m_1, m_2, \ldots, m_i)\), task \( \tau_i \) can be feasibly scheduled by the DM scheduling if

\[
\sum_{j=1}^{i} C_{j}^{\theta_j(m_j)} + D_i \cdot \sum_{j=1}^{i} U_{j}^{\theta_j(m_j)} \leq D_i
\]

\[
\Rightarrow \left( \sum_{j=1}^{i-1} C_{j}^{\theta_j(m_j)} + D_{i-1} \cdot \sum_{j=1}^{i-1} U_{j}^{\theta_j(m_j)} \right)
\]

\[
+ \left( C_i^{\theta_i(m_i)} + D_i \cdot U_i^{\theta_j(m_j)} + (D_i - D_{i-1}) \sum_{j=1}^{i-1} U_{j}^{\theta_j(m_j)} \right) \leq D_i
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$$
\sum_{j=1}^{i} C_{j}^{\theta_j(m_j)} + D_i \cdot \sum_{j=1}^{i} U_{j}^{\theta_j(m_j)} \leq D_i
$$

Therefore, demand for the first $i - 1$ tasks at time $D_{i-1}$

$$
+ \left( C_{i}^{\theta_i(m_i)} + D_i \cdot U_{i}^{\theta_j(m_j)} + (D_i - D_{i-1}) \sum_{j=1}^{i-1} U_{j}^{\theta_j(m_j)} \right) \leq D_i
$$
Dynamic Programming

- What is the minimum cost to be feasible?
- What is the minimum cost to be feasible for the first $i$ tasks under the approximation?
  - Two terms matter: the (sub-)demand $D_i \cdot \sum_{j=1}^{i} U_j$ and (sub-)demand $\sum_{j=1}^{i} C_{j}^{\theta_{j}(m_{j})}$.

Suppose that $G(i, \delta, u)$ is the minimum system cost, represented by a version selection $m_1, m_2, \ldots, m_i$, for the first $i$ tasks such that

- the total utilization for the first $i$ tasks is equal to $u$,
- the total execution time for the first $i$ tasks is equal to $\delta \cdot D_i$, and
- $\sum_{j=1}^{k} C_{j}^{\theta_{j}(m_{j})} \frac{D_{k}}{D_{k}} + \sum_{j=1}^{k} U_{j}^{\theta_{j}(m_{j})} \leq 1$ for any $k = 1, 2, \ldots, i$. 
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- the total execution time for the first \(i\) tasks is equal to \(\delta \cdot D_i\), and
- \(\sum_{j=1}^{k} \frac{C_j^{\theta_j(m_j)}}{D_k} + \sum_{j=1}^{k} U_j^{\theta_j(m_j)} \leq 1\) for any \(k = 1, 2, \ldots, i\).
Constructing $G(i, \delta, u)$ can be done by using the standard dynamic programming approach.

- Details [tighter definition and recursion] are in the paper
- The minimum $G(N, \delta, u)$ for $0 \leq \delta \leq 1$ and $0 \leq u \leq 1$ has a $(1,2)$-approximation factor for $N$ tasks.
- The solution is optimal on a slow-down platform with speed $\frac{1}{2}$.
- It takes pseudo-polynomial time/space for building the table.

Instead of building $G(i, \delta, u)$ for all possible values of $\delta$ and $u$:

- Approximate the values of interests
- Build the table by a user-specified granularity $\sigma$
- Lose some accuracy but earn the efficiency
- $(1, \frac{2}{1-\eta})$-approximation with time complexity $O((\frac{N}{\eta})^2 \sum_{i=1}^{N} w_i)$ under the DM scheduling by setting $\sigma$ to $\frac{1}{\left\lceil \frac{3N}{\eta} \right\rceil}$ for any given $\eta$ with $0 < \eta < 1$. 
Dynamic Programming (cont.)

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- Constructing $G(i, \delta, u)$ can be done by using the standard dynamic programming approach.
  - Details [tighter definition and recursion] are in the paper
  - The minimum $G(N, \delta, u)$ for $0 \leq \delta \leq 1$ and $0 \leq u \leq 1$ has a $(1,2)$-approximation factor for $N$ tasks.
  - The solution is optimal on a slow-down platform with speed $\frac{1}{2}$.
- It takes pseudo-polynomial time/space for building the table

**EDF**

The some design philosophy also works for EDF scheduling (for arbitrary deadlines) with some minor changes.

- Build the table by a user-specified granularity $\sigma$
- Lose some accuracy but earn the efficiency
- $(1, \frac{2}{1-\eta})$-approximation with time complexity $O((\frac{N}{\eta})^2 \sum_{i=1}^{N} w_i)$ under the DM scheduling by setting $\sigma$ to $\frac{1}{\left\lfloor \frac{3N}{\eta} \right\rfloor}$ for any given $\eta$ with $0 < \eta < 1$
No algorithm with finite approximation factors.
No algorithm with finite approximation factors

Algorithms with constant approximation factors

\( \mathcal{APX} \)
Classification of $\mathcal{NP}$-Hard Problems

No algorithm with finite approximation factors

Algorithms with constant approximation factors

PTAS

$\mathcal{APX}$

**PTAS (Polynomial-time Approximation Scheme):** For each constant $\epsilon > 0$, a polynomial-time partitioning algorithm $A_\epsilon$, with approximation factor $(1 + \epsilon)$.

- complexity depends on $\frac{1}{\epsilon}$, which is assumed to be a constant, e.g., $O(n^{\frac{2}{\epsilon}})$
- allows for a trade-off of run-time versus accuracy
d-Dimensional Representative Vector Set (Chen and Chakraborty, ECRTS’12)

Among \( t \in (0, \infty] \), choose \( t_1, \ldots, t_d \) for density values \( \frac{dbf(\tau_i, t_j)}{t_j} \) for \( j = 1, \ldots, d \).

- Representative: For the accuracy
- Constant dimensions: For complexity

A \( d \)-dimensional representative vector set \( \mathcal{V} \) for the given task set \( \mathcal{T} \) under a user-defined tolerable error \( 0 < \eta \):

- for any configuration \( \mathcal{T} \) and the corresponding vector set \( \mathcal{V} \)

\[
\gamma(\mathcal{T}) \geq \max_{j=1,2,\ldots,d} \sum_{v_i \in \mathcal{V} \quad q_{i,j}} \geq \left( \frac{1}{1 + \eta} \right) \gamma(\mathcal{T}),
\]

Less sampling points \hspace{1cm} Bounded error

where \( \gamma(\mathcal{T}) \) is the maximum density of \( \mathcal{T} \).
Among \( t \in (0, \infty] \), choose \( t_1, \ldots, t_d \) for density values \( \frac{\text{dbf}(\tau_i, t_j)}{t_j} \) for \( j = 1, \ldots, d \).

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\]

where \( \gamma(\mathcal{T}) \) is the maximum density of \( \mathcal{T} \).
(1 + \epsilon, 1 + \eta)-Approximation for EDF

- Chen and Chakraborty, ECRTS’12: when $\frac{D_{\text{max}}}{D_{\text{min}}}$ is a constant, the number of representative dimensions is a constant.
- How do we achieve (1 + \epsilon, 1 + \eta)-Approximation?
  - Build a d-dimensional representative vector set $\mathcal{V}$ for 1 + \eta speed-up guarantee
  - The problem is reduced to a minimum-cost multiple choice multiple-dimension knapsack problem
    - Set $Z = \min\{N, \lceil \frac{d}{\epsilon} \rceil \}$, bounded by a constant
    - Enumerate the combinations to select the versions for $Z$ large tasks
    - Select the versions of the other $N - Z$ light tasks by using linear programming
    - Round the fractional variables to yield a feasible solution
    - Return the best found feasible solution as the result
(\(\alpha, \beta\))-approximation for combinatorial optimization problems in RTS

- With respect to speeding-up: the platform is speeded up to \(\beta\) to ensure the feasibility and optimality.
- With respect to slowing-down: the derived solution is \(\alpha\)-approximation with respect to a problem instance by slowing down the platform to \(\frac{1}{\beta}\).

<table>
<thead>
<tr>
<th></th>
<th>EDF</th>
<th>DM ((D_i \leq T_i))</th>
</tr>
</thead>
<tbody>
<tr>
<td>pseudo-polynomial</td>
<td>(1,1.6322)</td>
<td>(1,2)</td>
</tr>
<tr>
<td>polynomial</td>
<td>(1, (\frac{1.6322}{1-\eta}))</td>
<td>(1, (\frac{2}{1-\eta}))</td>
</tr>
</tbody>
</table>
| polynomial     | (\(\frac{1}{1-\epsilon}\), \(\frac{1}{1-\eta}\)) | ????????

\(\frac{D_{max}}{D_{min}}\) is a constant