Polynomial-Time Exact Schedulability Tests for Harmonic Real-Time Tasks

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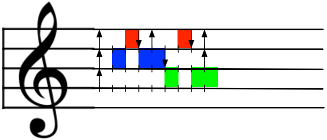


Outline

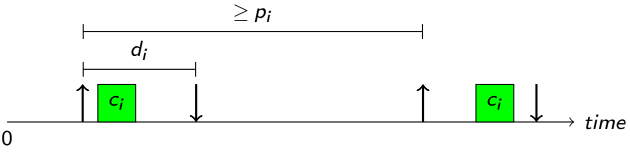
Sporadic Task Model – Constrained Deadlines

Tasks $\tau_1, \tau_2, \ldots, \tau_n$

Task τ_i has:

- a worst-case computation time c_i
- a relative deadline d_i
- a minimum inter-arrival time (period) p_i

Example:



We assume all parameters integral and constrained deadlines ($d_i \leq p_i$)

Scheduling Model

- One processor (so $\sum_i c_i/p_i \leq 1$)
- Preemptions allowed
- No preemption overheads

Preemptive Scheduling Policies Considered

- Fixed Priority (FP)
  - task-level priority
  - give priority to the tasks according to a static ordering (p_i: Rate Monotonic, d_i: Deadline Monotonic, ...)
  - jobs inherit priority from the originating task
- Earliest Deadline First (EDF)
  - job-level priority
  - give priority to the jobs with earliest absolute deadlines

In both cases, at any time step, schedule the available job with highest priority

Schedulability Testing

A task system is ALG-schedulable if, for every of its (legal) job sequences, algorithm ALG constructs a feasible schedule

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Let $P = \max_i p_i$, $n =$ number of tasks

A schedulability test runs in

- pseudopolynomial time if it runs in time $O(n^a P^b)$ for some a, b
- polynomial time if it runs in time $O(n^a (\log P)^b)$ for some a, b

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Remark. Enough to analyze the Synchronous Arrival Sequence:

- jobs of τ_i released at 0, $1 \cdot p_i$, $2 \cdot p_i$, $3 \cdot p_i$, $\ldots$

Schedulability Testing: FP

FP-schedulability is often verified via Response Time Analysis.

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Input: priority-ordered task system τ, integer R
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Response Time Analysis is (weakly) NP-hard.

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Our Results

For task sets with harmonic periods and constrained deadlines:

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Lehoczky, Sha & Ding (1989); Earlier folklore?

Rate Monotonic correctly schedules any harmonic task set with implicit deadlines and $U \leq 1$

However, we consider constrained deadlines and arbitrary priorities

FP-Schedulability
(Response Time Analysis)

Response Time Analysis: A Binary Search Approach

Recall that the response time r_n of τ_n is the least t such that:

$$c_n + \sum_{i<n} \left\lceil \frac{t}{p_i} \right\rceil \cdot c_i \leq t. \qquad (1)$$

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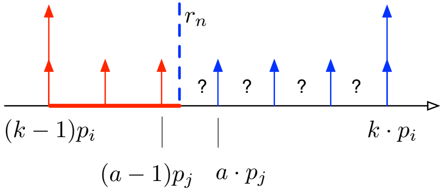
Idea: use binary search to look for r_n
Let p_j, p_i be "consecutive" periods (so $p_j \mid p_i$)
Say $r_n \in ((k-1)p_i, kp_i]$



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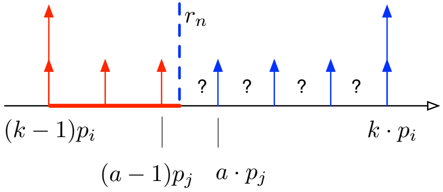

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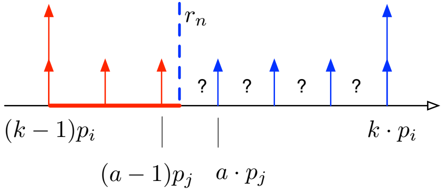

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Key Lemma

If periods are harmonic, then $t(a) := a \cdot p_j$ satisfies Condition (1) for any $a \in \mathbb{N}$ such that $t(a) \in [r_n, k \cdot p_i]$.

$\Rightarrow$ recurse on $((a-1)p_j, a \cdot p_j]$

Response Time Analysis: Algorithm and Summary

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Algorithm 1:

$LB \leftarrow 0$

$UB \leftarrow P \cdot c_n$ *[invariant: $LB < r_n \leq UB$]*

repeat $n-1$ times (for decreasing values of p_j):

  Binary search the least a such that $a \cdot p_j$ satisfies Condition (1)

  $LB \leftarrow (a-1) \cdot p_j$

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Theorem

Algorithm 1 is correct. Its running time is $O(n \log P)$.

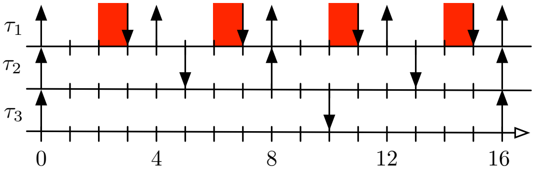
EDF-Schedulability

EDF-Schedulability: Procrastination

We analyze a non-EDF, yet optimal schedule: "Procrast" ("procrastinating")

We'll show Procrast-schedulability ≡ EDF-Schedulability

Let $p_1 \leq p_2 \leq \ldots \leq p_n$. Procrast schedule example:



Similar to a "backwards" Rate Monotonic schedule

Key Lemma

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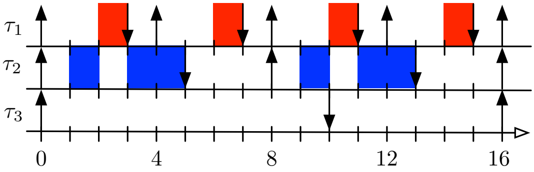

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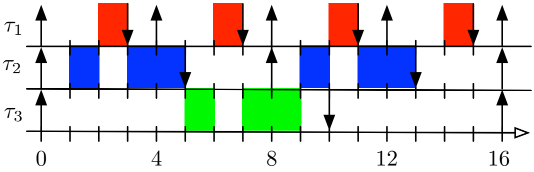

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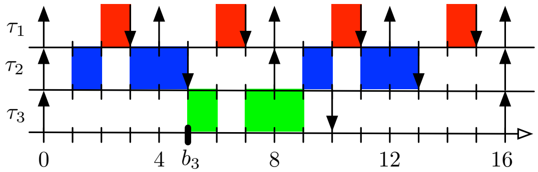
(1) Procrast-schedulability can be tested in polynomial time

Due to harmonicity, starting time of τ_j relative to p_j is a constant

Call it start offset (b_j). The b_j's compactly describe the schedule



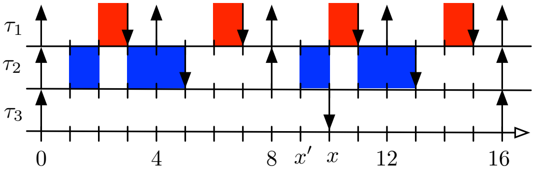
Easy to compute b_j efficiently if we have an efficient procedure for $IDLE_j(x) :=$ idle time in $[x, d_j)$ (after fixing $\tau_1, \ldots, \tau_{j-1}$)

$IDLE_j$ allows to compute b_j efficiently via binary search

Computing $IDLE_j(x)$

$IDLE_j(x) :=$ idle time in $[x, d_j)$ (after fixing $\tau_1, \ldots, \tau_{j-1}$)

Compute largest $x' \leq x$ such that x' is not beyond the start offset of smaller period tasks (can be done recursively)



- During $[x', x)$ the processor must be busy (by construction)
- During $[0, x')$ the total work C is on jobs already completed at x' $\Rightarrow$ easy to compute C

$IDLE'_j[0, x) = x' - C$.

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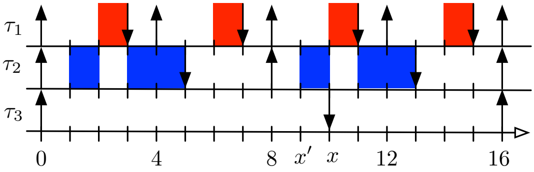

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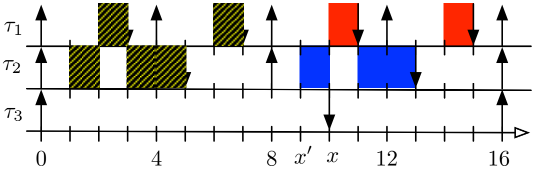

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(2) Procrast is optimal

Lemma

For each j and $x \geq 0$, the Procrast schedule for $\tau_1, \ldots, \tau_j$ maximizes the amount of idle time in $[0, x)$ (namely, no feasible schedule has more idle time).

Proof by induction. $j = 1$ holds by construction. □

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If Procrast construction fails
$\Rightarrow$ no schedule has c_j idle time within $[0, d_j)$
$\Rightarrow$ taskset is infeasible.

EDF-Schedulability: Algorithm and Summary

Algorithm 2:
Assume $p_1 \leq p_2 \leq \ldots \leq p_n$ (if not, reorder the tasks)
for $j = 1$ to n **do**:
   $idlemax \leftarrow IDLE_j(0)$
   if $idlemax < c_j$ **then return** infeasible
   else
      $b_j \leftarrow \max\{x : IDLE_j(x) = c_j\}$
return (feasible,$(b_1, b_2, \ldots, b_n)$)

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Theorem

Algorithm 2 is correct. Its running time is $O(n^3 \log P)$.

EDF-SCHEDULABILITY for "Jointly" Harmonic Tasksets

A taskset is jointly harmonic if for all $x_i, x_j \in \{d_1, \ldots, d_n, p_1, \ldots, p_n\}$, either $x_i \mid x_j$ or $x_j \mid x_i$.

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The absence of strictly crossing pairs simplifies the schedule considerably

Conclusions and Further Directions

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Open directions:

- extension to the arbitrary deadline case ($d_i \lesseqgtr p_i$)
- extension to multiprocessors – are the problems NP-hard or not?
- improve the $O(n^3 \log P)$ bound to $O(n \log P)$?